Unavoidable patterns in complete simple topological graphs

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(joint work with Andrew Suk)

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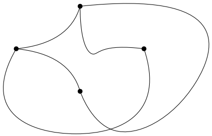
September 2022

Simple topological graph

Vertices = points in the plane

Edges = curves connecting the points (vertices)

Simple = any two curves (edges) have at most one intersection point, i.e. a common endpoint or a crossing.



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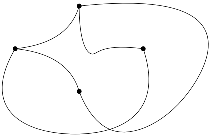
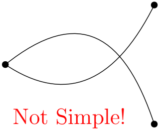
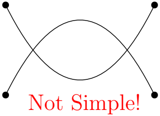
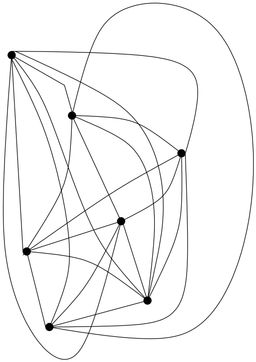






Ramsey-type Theme

What large patterns can we find in complete simple topological graphs?



Example: Non-crossing structures

Theorem (Suk 2013, Fulek–Ruiz-Vargas 2014)

Every n-vertex complete simple topological graph contains $\Omega(n^{\frac{1}{3}})$ pairwise disjoint edges.

Later bound: $n^{\frac{1}{2}-o(1)}$ by Ruiz-Vargas 2015; $\Omega(n^{\frac{1}{2}})$ by Aichholzer et al. 2022.

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Theorem (Pach–Solymosi–Tóth 2003)

Every n-vertex complete simple topological graph contains a non-crossing path of length $\Omega((\log n)^{\frac{1}{6}})$.

New bound: $(\log n)^{1-o(1)}$ by Aichholzer et al. 2022 and Suk-Z. 2022 indepedndently.

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Fact (Rafla 1988, Ábrego et al. 2015)

Every complete simple topological graph with at most 9 vertices contains a non-crossing Hamiltonian cycle.

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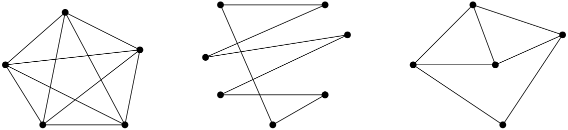
Conjecture (Rafla 1988)

Every complete simple topological graph ~~with at most 9 vertices~~ contains a non-crossing Hamiltonian cycle.

Vertices = plane points in general position, i.e. no collinear triples

Edges = straight lines connecting the points (vertices)



Example: Geometric graph

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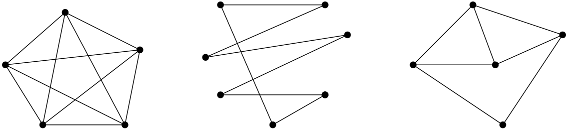

Convex = points (vertices) in convex position

Example: Geometric graph

Question

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Theorem (Erdős–Szekeres 1935)

Every set of $\binom{2m-4}{m-2}+1$ *plane points in general position contains a subset of* m *elements in convex position.*

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What large patterns can we find in complete geometric graphs?

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Corollary

Every n*-vertex complete geometric graph contains a* m*-vertex complete convex geometric graph* C_m *with* $m = \Omega(\log n)$*.*

Avoiding C_5 in topological graphs?

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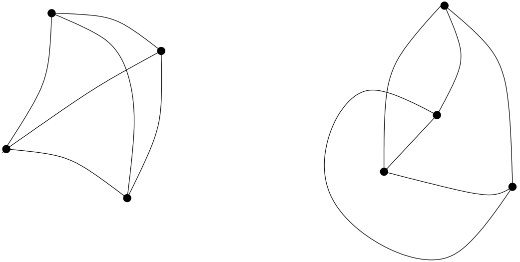
Definition

Topological graphs G and H are **weakly isomorphic** if there is a graph-theoretic isomorphism between them such that two edges in G cross if and only if the corresponding edges in H cross.



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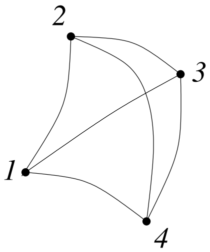

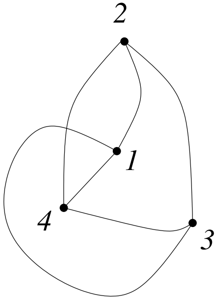

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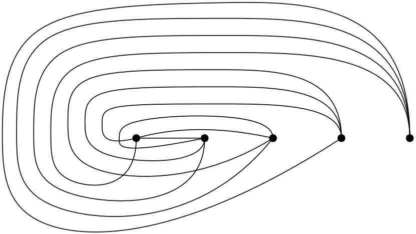

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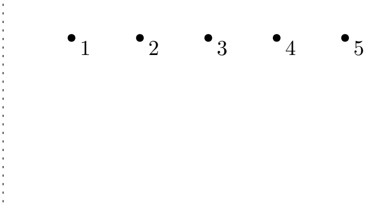

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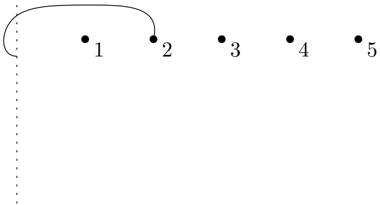

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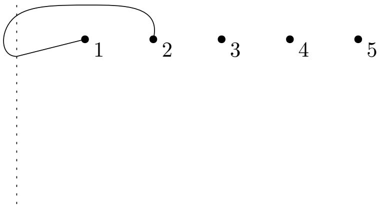

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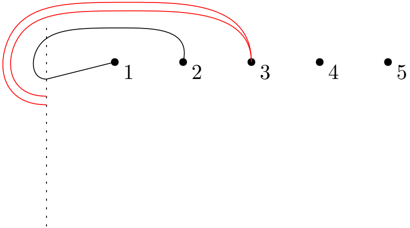

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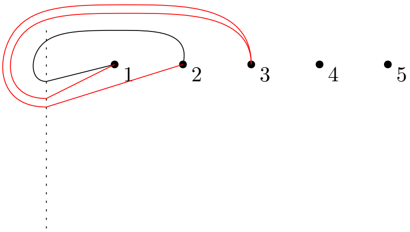

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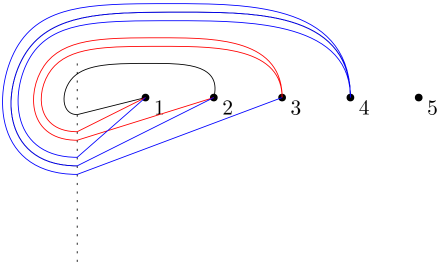

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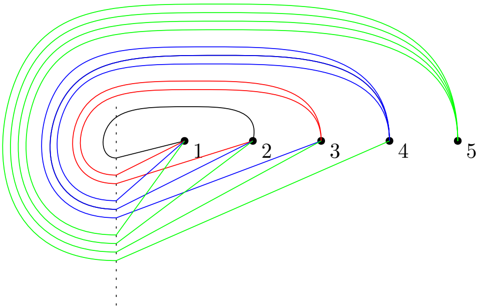
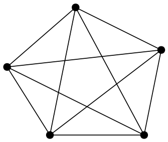


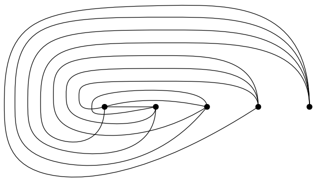
Main result

However, we can't avoid both C_5 and T_5.

Theorem (Pach–Solymosi–Tóth 2003)

Every n-vertex complete simple topological graph contains a topological subgraph on $m \geq \Omega((\log n)^{\frac{1}{8}})$ vertices that is weakly isomorphic to C_m or T_m.





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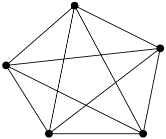
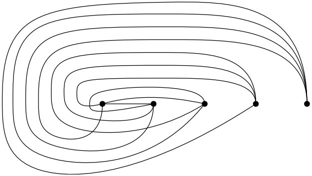
Theorem (Suk–Z. 2022)

Every n-vertex complete simple topological graph has a topological subgraph on $m \geq (\log n)^{\frac{1}{4}-o(1)}$ *vertices that is weakly isomorphic to* C_m *or* T_m.





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We also have long non-crossing path.

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Set-up

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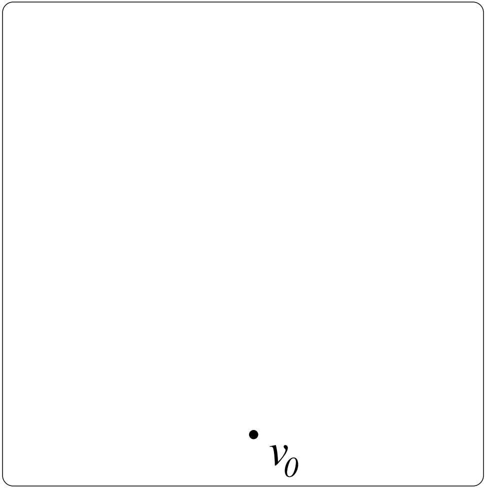

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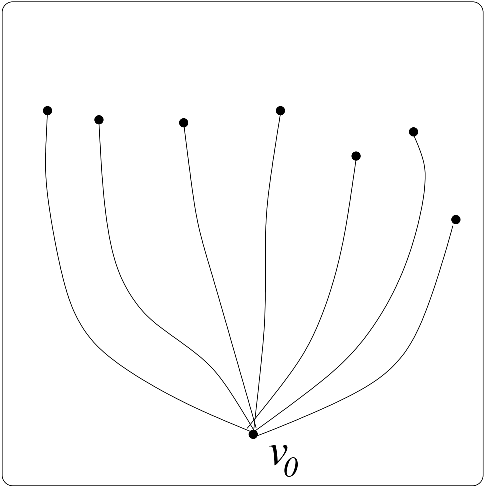
$K_n =$ 

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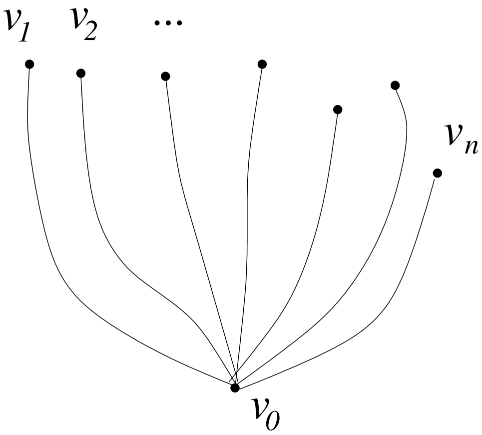

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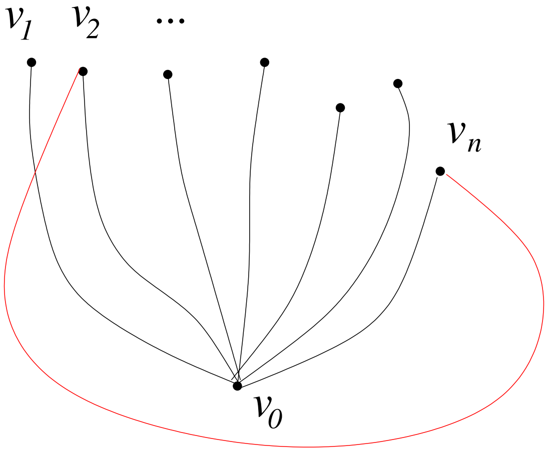

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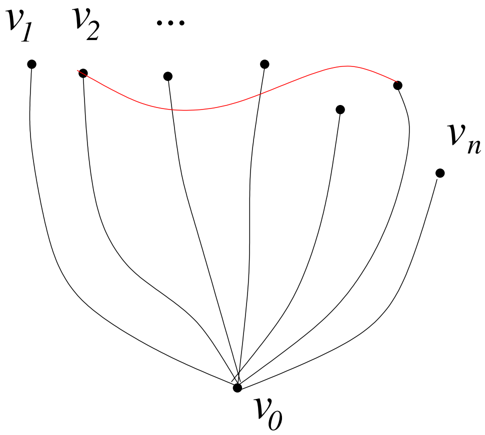
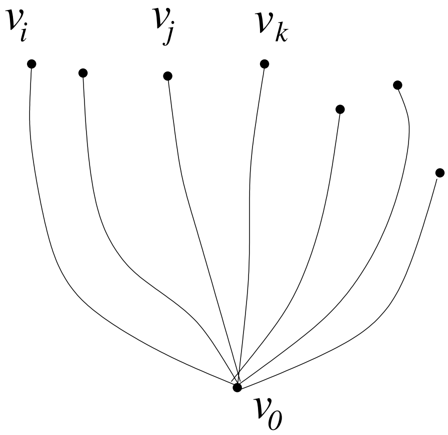


Coloring triples

Observation (Pach–Solymosi–Tóth)

For $v_i < v_j < v_k$, there are only 4 configurations.



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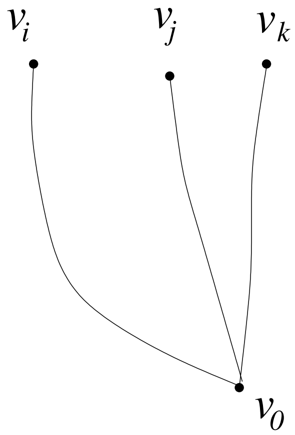

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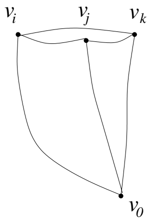

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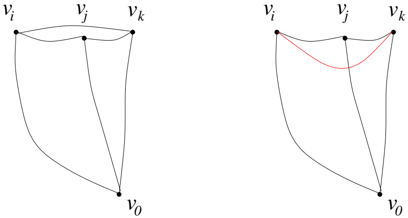

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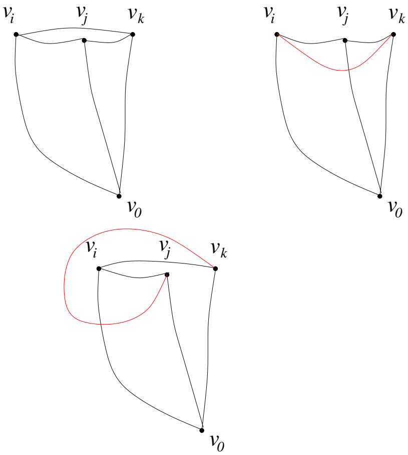

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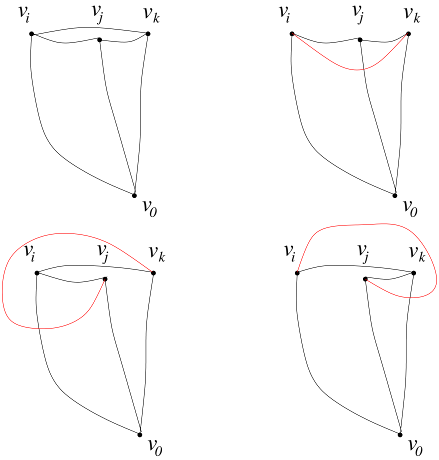
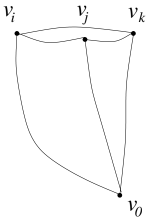
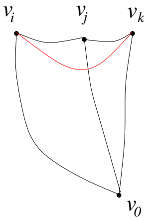


Pach–Solymosi–Tóth: Color the triple (v_i, v_j, v_k) using $\{000, 010, 100, 001\}$

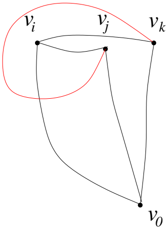
000



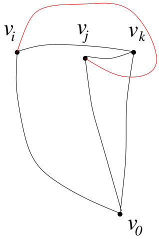
010



100

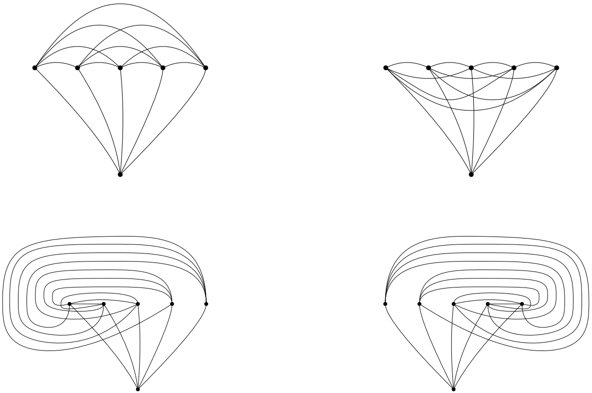


001



Coloring triples

Fact: If there are m vertices with all triples monochromatic, then they form a weakly-isomorphic copy of C_m or T_m.

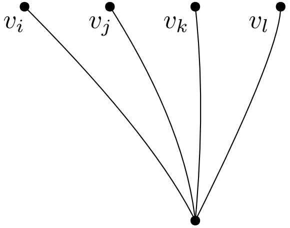


Transitive colors: 100 and 001

Observation

The colors 100 and 001 are transitive.

For $v_i < v_j < v_k < v_\ell$, if (v_i, v_j, v_k) and (v_j, v_k, v_ℓ) have color 001, then so does (v_i, v_j, v_ℓ) and (v_i, v_k, v_ℓ).



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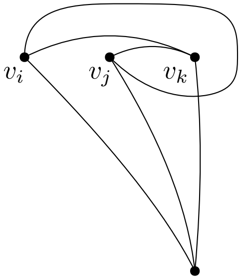

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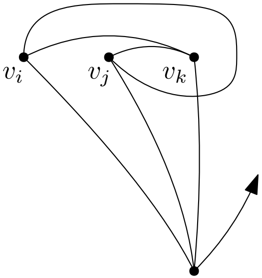

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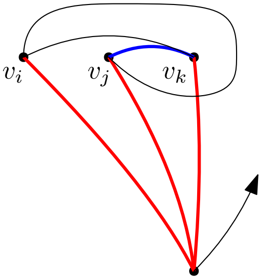

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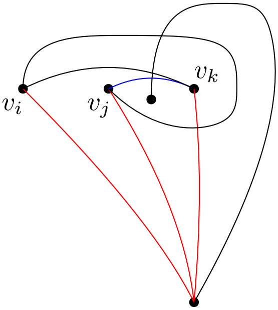

Monotone path

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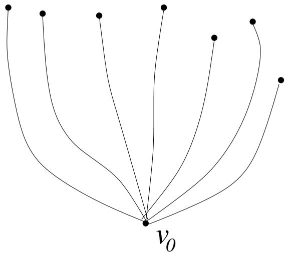
Monochromatic monotone path: vertices $u_1 < u_2 < \cdots < u_m$ all triples (u_i, u_{i+1}, u_{i+2}) monochromatic.



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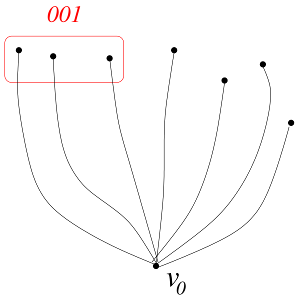

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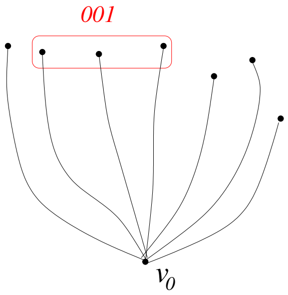

Monotone path

Observation

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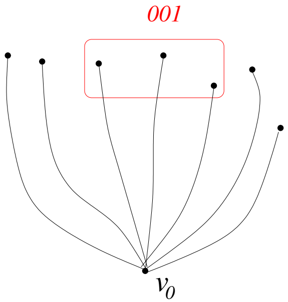

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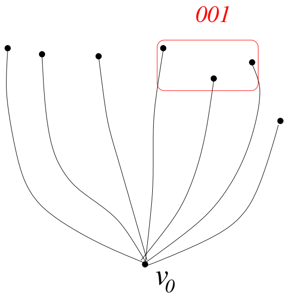

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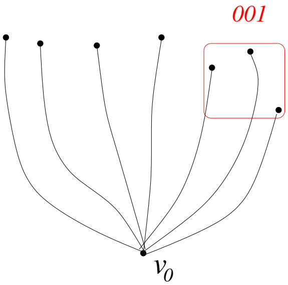

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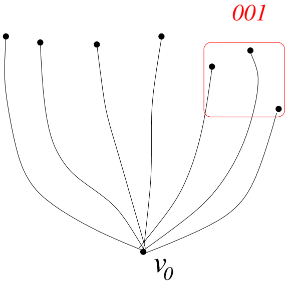
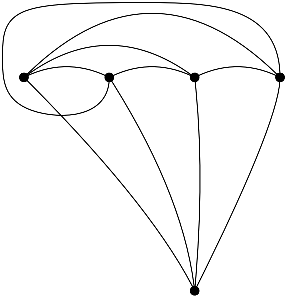
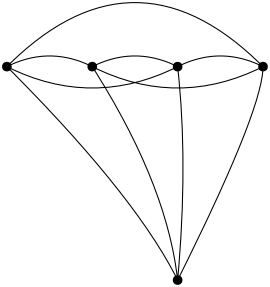


Corollary

A mono-χ monotone path of length m in color 100 or 001 is a mono-χ clique.

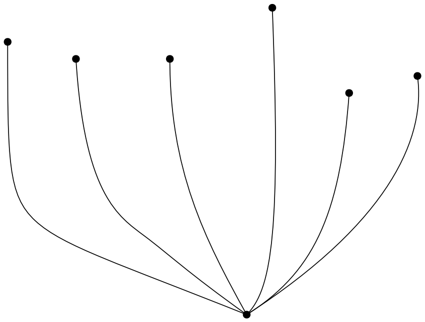
However, 000 and 010 are not transitive.





Forward path

Monochromatic forward path: vertices $u_1 < u_2 < \cdots < u_m$ such that all triples (u_i, u_{i+1}, u_j) are monochromatic.



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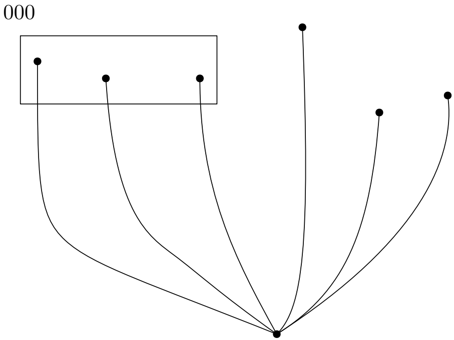

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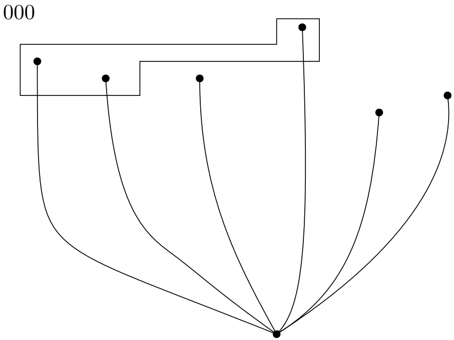

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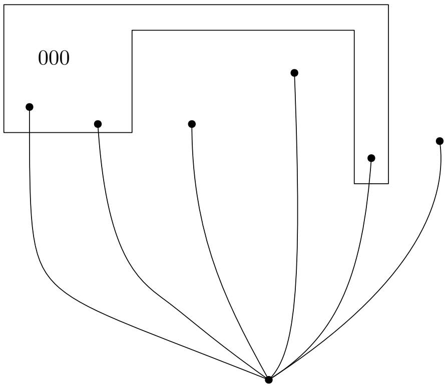

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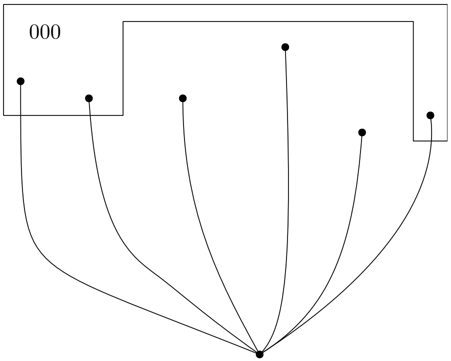

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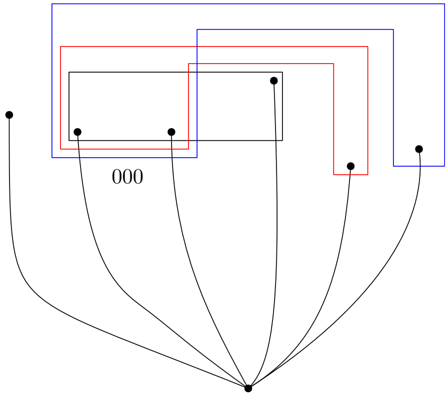

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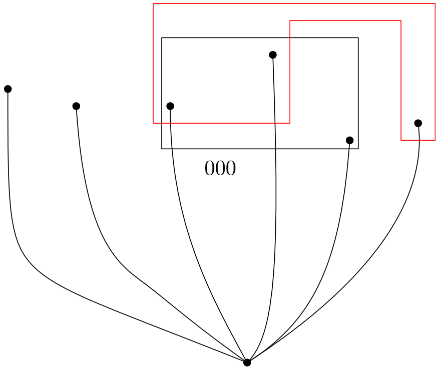

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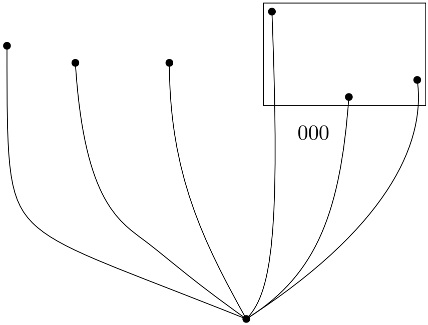

000 and 010

Lemma (essentially Pach–Solymosi–Tóth 2003)

If there are vertices $u_1 < u_2 < \cdots < u_m$ with all triples (u_i, u_j, u_k) in color 000 or 010, and forming a mono-χ forward path of length m, then they form a mono-χ clique.

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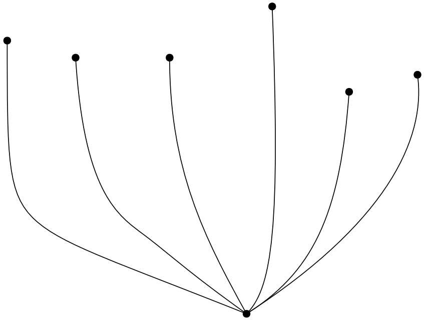

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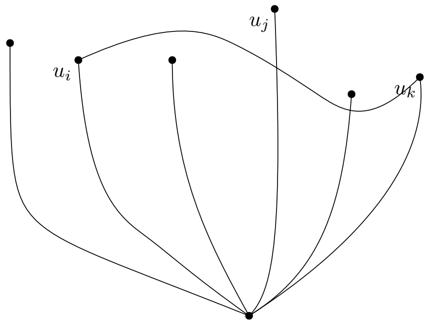

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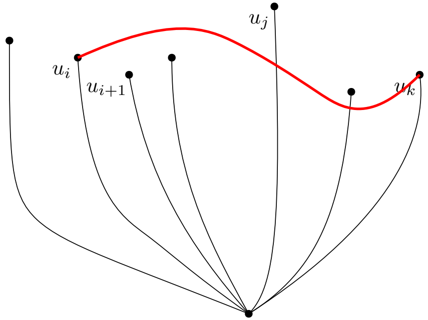

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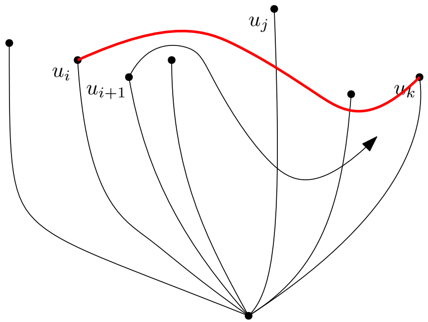

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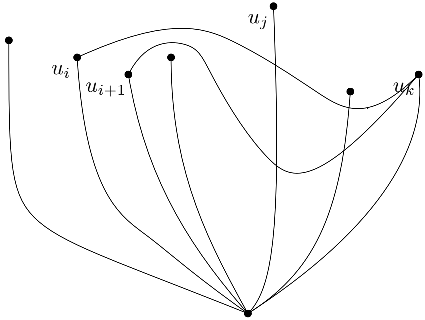

Combinatorial statement

Theorem

Every coloring of all triples of $[n]$*, where* $n = 2^{O(m^4(\log m)^2)}$*, by red, blue, green, and yellow contains*

- *a subset with only red or blue triples, and forming a mono-*χ *forward path of length m; OR*
- *a mono-*χ *monotone path of length m in green or yellow.*

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- *a subset with only red or blue triples, and forming a mono-χ forward path of length m; OR*
- *a mono-χ monotone path of length m in green or yellow.*

Letting red=000, blue=010, green=100, and yellow=001, this implies our theorem of unavoidable patterns.

Erdős–Szekeres-type results

Theorem (essentially Erdős–Szekeres 1935)

Let $f(m)$ be the minimum n such that every 2-coloring of all triples of $[n]$ contains a mono-χ monotone path of length m. We have $f(m) = \binom{2m-4}{m-2} + 1$.

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Theorem (Fox–Pach–Sudakov–Suk 2012)

Every q-coloring of all triples of $[n]$, where $n = 2^{O(m^q \log m)}$, contains a mono-χ forward path of length m.

- Fox–Pach–Sudakov–Suk stated this result for monotone paths.
- The proof uses optimal strategies of online Ramsey games.

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Our combinatorial statement can be proved by combining ideas from above theorems.

Non-crossing path

Theorem (Aichholzer et al. 2022; Suk-Z. 2022)

Every n-vertex complete simple topological graph contains a non-crossing path of length $(\log n)^{1-o(1)}$.

Proof.

Non-crossing path

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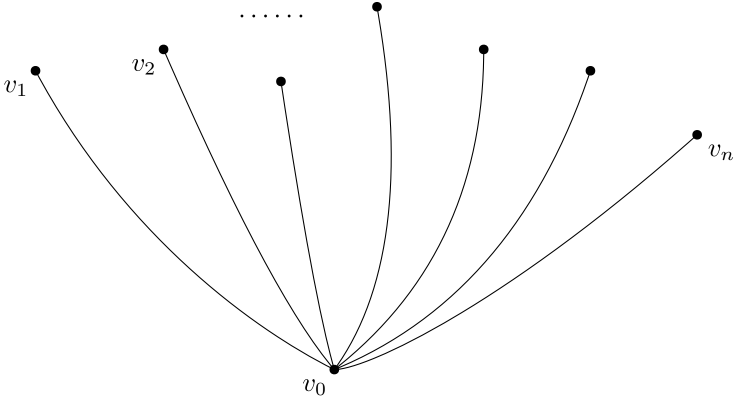


We consider the sequence of curves emanating from v_1 in counterclockwise order.

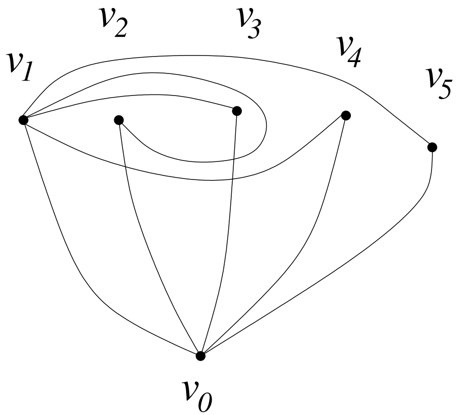
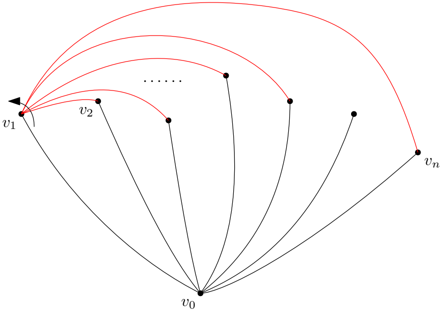


Figure: $(v_1v_4, v_1v_3, v_1v_2, v_1v_5)$

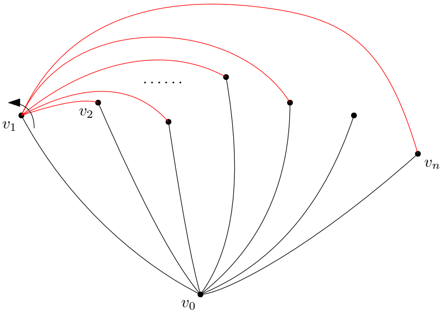
Non-crossing path

Case 1: Non-crossing $K_{2,m}$ with $m = (\log n)^2$.



Increasing sequence of length m.

Case 1: Non-crossing $K_{2,m}$ with $m = (\log n)^2$.

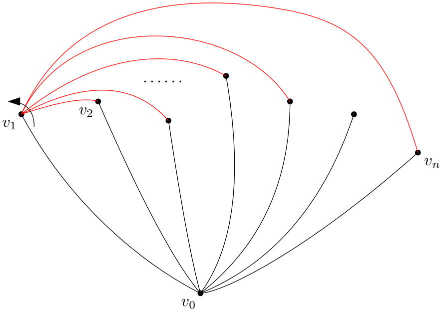


Lemma (Fulek–Ruiz-Vargas 2015)

Inside a complete simple topological graph, the induced subgraph on the m-part of a non-crossing $K_{2,m}$ contains a dense subgraph weakly isomorphic to a x-monotone topological graph.

Non-crossing path

Case 1: Non-crossing $K_{2,m}$ with $m = (\log n)^2$.

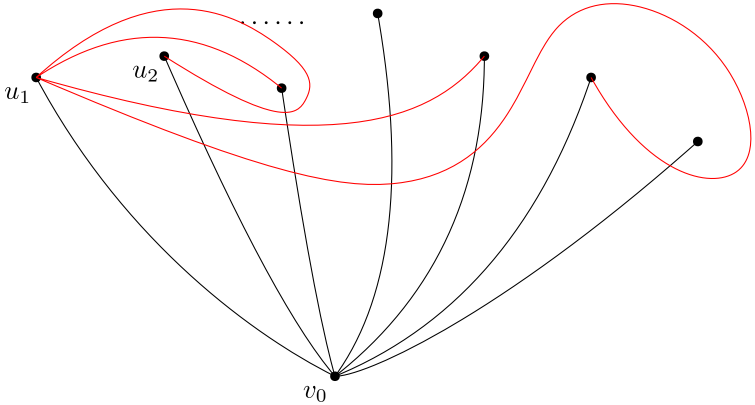


Lemma (essentially Tóth 2000)

Every dense x-monotone simple topological graph on m vertices contains a non-crossing path of length $\Omega(\sqrt{m})$.

Non-crossing path

Case 2: No non-crossing $K_{2,m}$ with $m = (\log n)^2$.



Decreasing sequence of length n/m.

Non-crossing path

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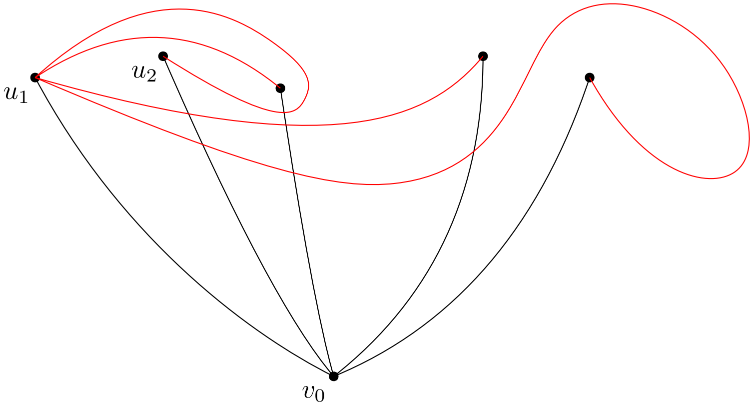
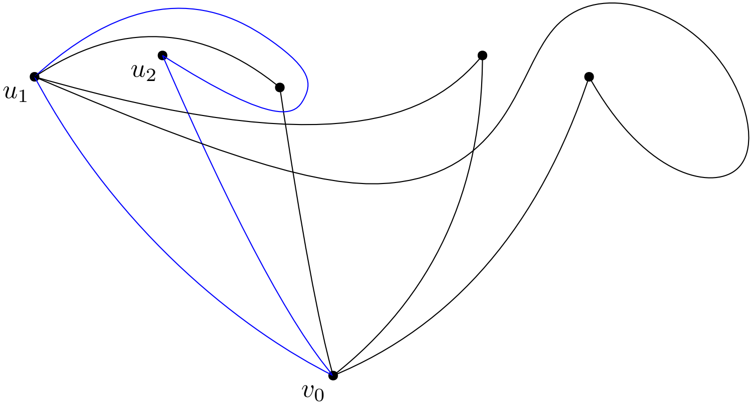


Keep only the decreasing sequence.

Non-crossing path

Case 2: No non-crossing $K_{2,m}$ with $m = (\log n)^2$.



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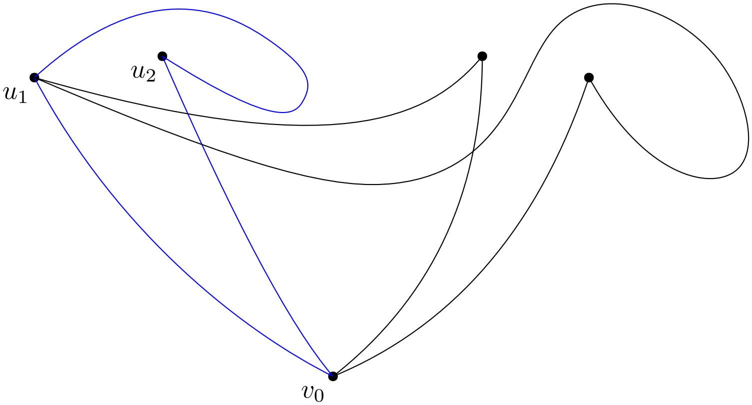

Keep only the vertices inside or outside the blue triangle.

Non-crossing path

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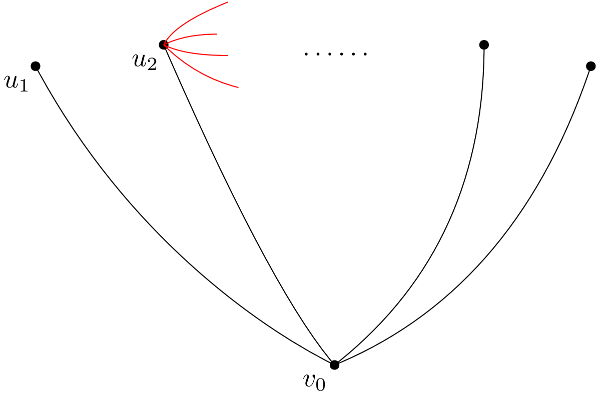
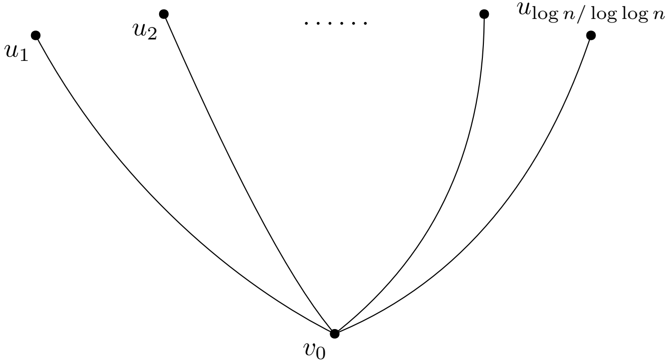


Repeat at next vertex.

Non-crossing path

Case 2: A path $u_1, u_2, \ldots, u_l$ of length $(\log n)^{1-o(1)}$.



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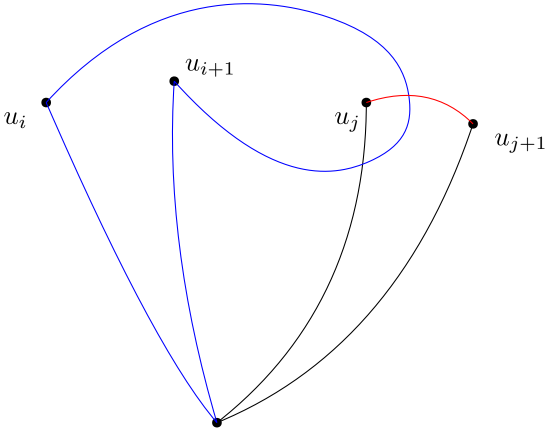

u_j and u_{j+1} can't be separated by the triangle $v_0 u_i u_{i+1}$.

Non-crossing path

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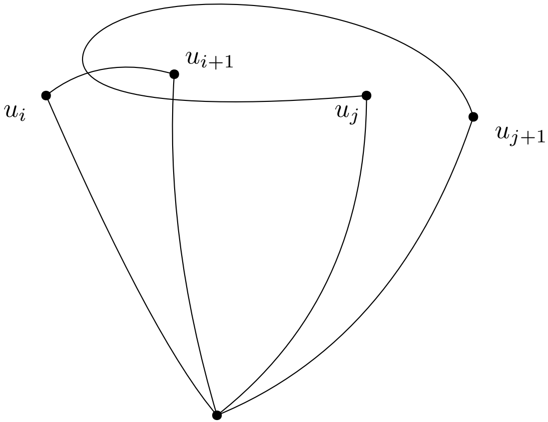

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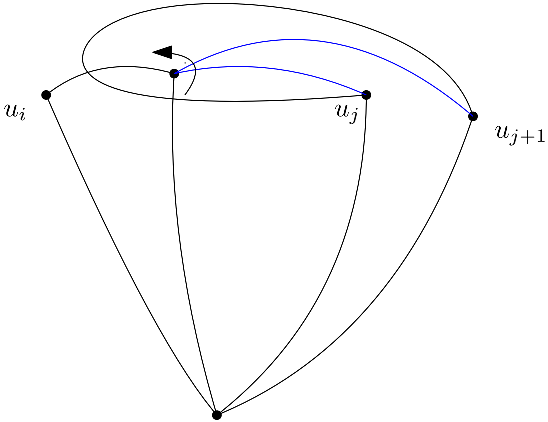

$u_{i+1}u_j$ and $u_{i+1}u_{j+1}$ can't be increasing.

Ramsey number?

Monochromatic forward path: vertices $u_1 < u_2 < \cdots < u_m$ such that all triples (u_i, u_{i+1}, u_j) are monochromatic.

Let $g(m)$ be the minimum n such that every 2-coloring of all triples of $[n]$ contains a mono-χ forward path of length m.

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Monochromatic backward path: vertices $u_1 < u_2 < \cdots < u_m$ s.t. all triples (u_i, u_j, u_{j+1}) are monochromatic.

Let $h(m)$ be the minimum n such that every 2-coloring of all triples of $[n]$ contains a mono-χ forward or backward path of length m.

Ramsey number?

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Let $h(m)$ be the minimum n such that every 2-coloring of all triples of $[n]$ contains a mono-χ forward or backward path of length m.

We know $2^{\Omega(m)} \leq h(m) \leq g(m) \leq 2^{O(m^2 \log m)}$.

Problem

Find better bounds for $g(m)$ and $h(m)$.

Upper bound?

What's the size of the largest weakly-isomorphic copy of C_m or T_m inside every n-vertex complete simple topological graph?
Lower bound: $(\log n)^{\frac{1}{4}-o(1)}$.

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Construction 1: Take n points in the plane with no $2\lceil \log n \rceil$ points in convex position (cf. Erdős–Szekeres 1935), and connect them using straight lines.

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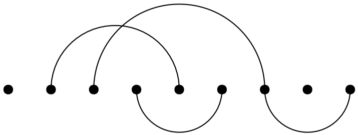

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Problem

Find better upper bound constructions.

Conclusion

Theorem (Suk–Z. 2022)

Every n-vertex complete simple topological graph has a topological subgraph on $m \geq (\log n)^{\frac{1}{4}-o(1)}$ vertices that is weakly isomorphic to C_m or T_m.

- Reduction to a problem of monotone and forward paths.
- Arguments from Fox-Pach-Sudakov-Suk 2012.

Theorem (Aichholzer et al. 2022; Suk-Z. 2022)

Every n-vertex complete simple topological graph contains a non-crossing path of length $(\log n)^{1-o(1)}$.

- Rigidity of non-crossing $K_{2,m}$ and a greedy argument.

Thank you!!!

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