Disjoint faces in simple topological graphs

Ji Zeng

Department of Mathematics
University of California San Diego

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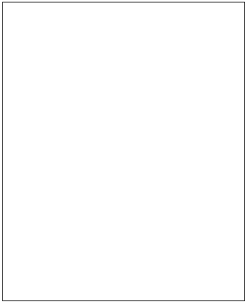
Place n points in the unit square. And consider the triangles.



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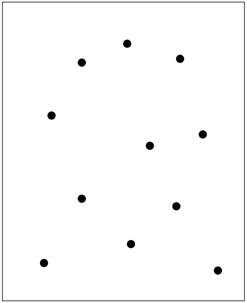

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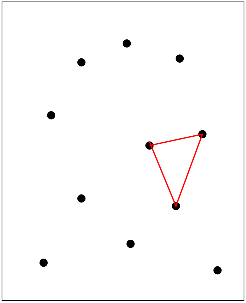

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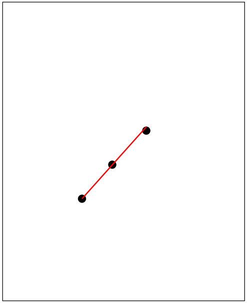

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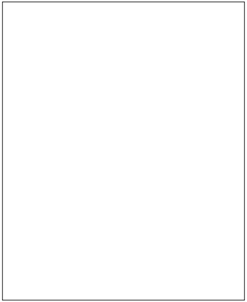


Can I make all the triangles to be small? Trivial, with or without collinear triples.

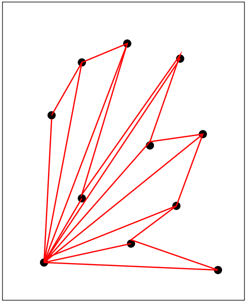
Heilbronn's triangle problem

Place n points in the unit square. And consider the triangles.

Can I make all the triangles to be small? Trivial, with or without collinear triples.

Can I make all the triangles to be large? Non-trival, Heilbronn's triangle problem.

Heilbronn's triangle problem: simple upper bound



There are $\Omega(n)$ disjoint triangles. Some triangle has area $O(1/n)$.

Heilbronn's triangle problem

Problem (Heilbronn)

What is the asymptotic growth rate of $h(n)$, the area of the smallest triangle determined by n points in the unit square, when these points are chosen to maximize this area?

Komlós-Pintz-Szemerédi 1981: $h(n) = O\left(\frac{1}{n^{\frac{8}{7}-\epsilon}}\right)$

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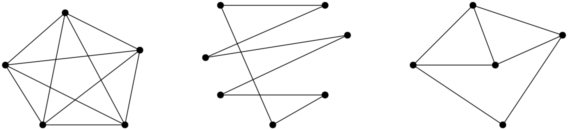
Cohen-Pohoata-Zakharov 2023+: $h(n) = O\left(\frac{1}{n^{\frac{8}{7}+\frac{1}{2000}}}\right)$

Geometric graph

Vertices = points in the plane in general position.
Edges = straight line segments connecting the vertices.



Problem (Heilbronn)

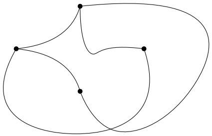
What is the asymptotic growth rate of $h(n)$, the area of the smallest triangle generated by a n-vertex complete geometric graph drawn in the unit square, when the drawing maximizes this area?

Simple topological graph

Vertices = points in the plane.

Edges = curves connecting the vertices.

Simple = any two edges intersect at most once, i.e. a common endpoint or a crossing.



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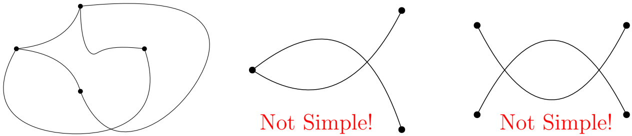
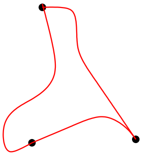
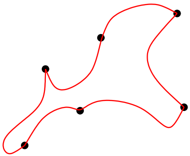


Faces in topological graph

k-**face:** Open bounded cell enclosed by a plane *k*-cycle.





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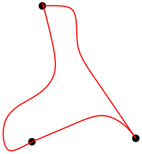
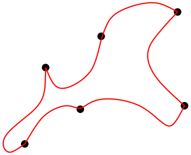
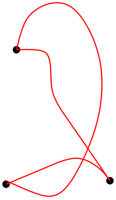
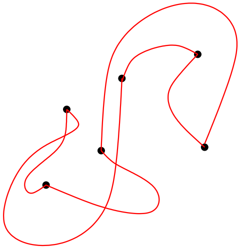




Not *k*-faces:



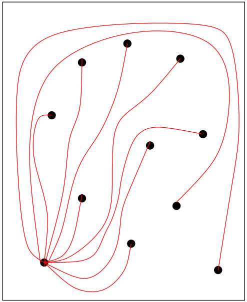


Topological Heilbronn's problem

We consider a topological variant of Heilbronn's problem.

Problem

What is the asymptotic growth rate of $\tilde{h}(n)$, the area of the smallest <u>3-face</u> generated by a n-vertex complete <u>simple topological graph</u> drawn in the unit square, when the drawing maximizes this area?



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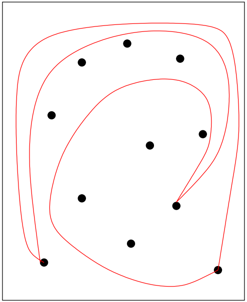
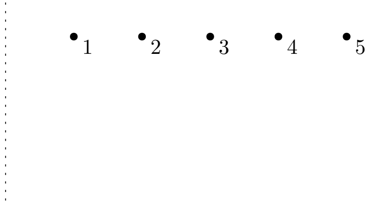


Complete twisted graph

Observation (Hubard–Suk 2023)

For any $\epsilon > 0$, we have $1 - \epsilon < \tilde{h}(n) < 1$.

Construction: complete twisted graph T_n (Harborth–Mengersen 1992)



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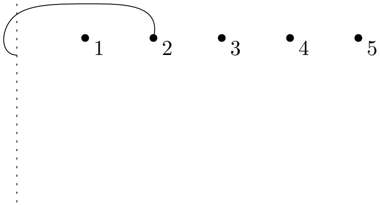

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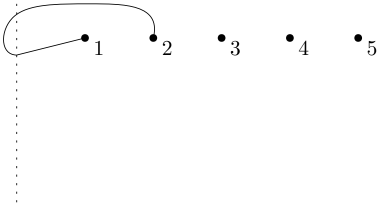

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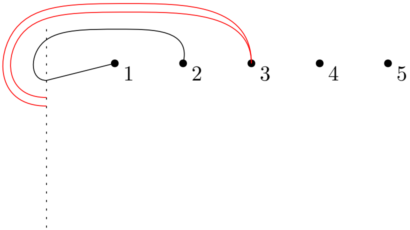

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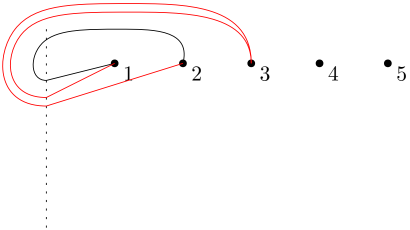

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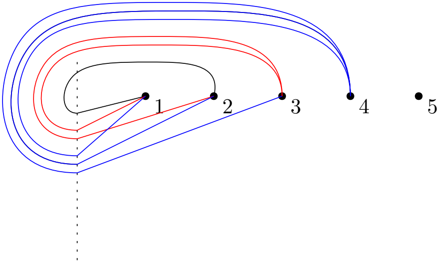

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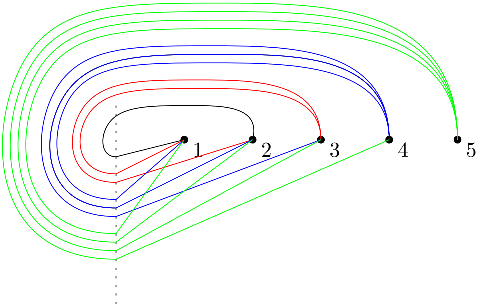

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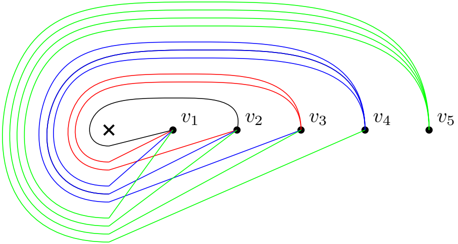

Every triangle of T_n contains the cross $\times$ in the illustration above.

Topological Heilbronn's problem for 4-faces

Problem

What is the asymptotic growth rate of $\tilde{h}_4(n)$, the area of the smallest 4-face generated by a n-vertex complete simple topological graph drawn in the unit square, when the drawing maximizes this area?

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Corollary (Hubard–Suk 2023)

$\tilde{h}_4(n) \leq O(1/n^{1/3})$.

Topological Heilbronn's problem for 4-faces

Theorem (Hubard–Suk 2023)

Every n-vertex complete simple topological graph generates at least $\Omega(n^{1/3})$ pairwise disjoint 4-faces.

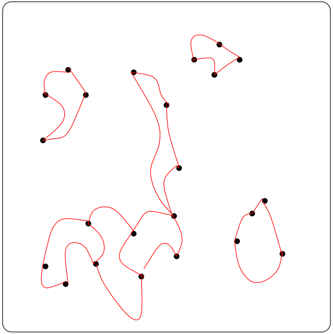
$K_n =$ 

Figure: Disjoint 4-faces can share boundary vertices or edges.

Main Result

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For every $n \geq 1$, there is a n-vertex complete simple topological graph drawn in the unit square such that every face it generates has area at least $\Omega(1/n)$.

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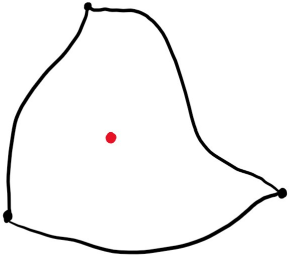
Theorem (Z. 2023)

For even $k \geq 4$, every n-vertex complete simple topological graph generates at least $(\log n)^{1/4-o(1)}$ pairwise disjoint k-faces.

Key phenomenon

Claim: If I complete this to a simple topological K_4, then at least two new edges won't cross the triangle.



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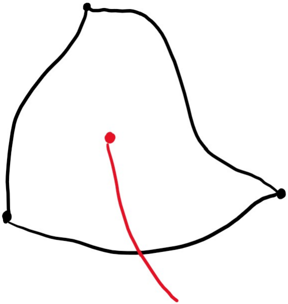

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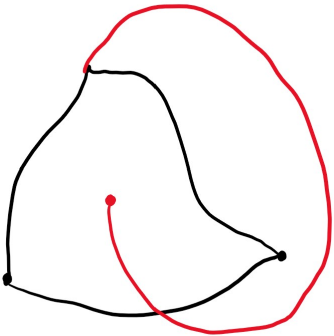

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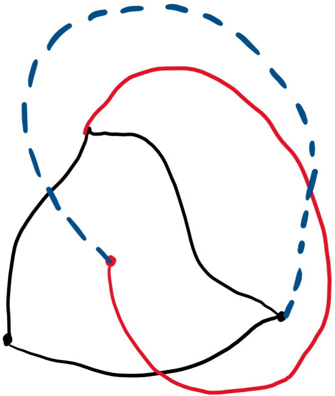
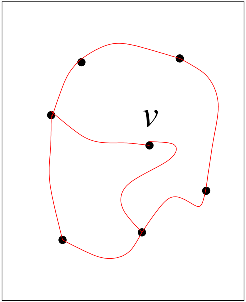


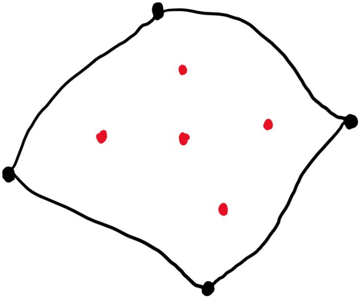
Key phenomenon

Lemma (Fulek–Ruiz-Vargas 2013)

In a complete simple topological graph, suppose H is a plane connected subgraph and v is a vertex not in H, then there exist at least two edges between v and the vertices of H that do not cross H.

$K_n =$ 

Claim: If I complete this to a simple topological K_9, then there is a new 4-face inside this already-drawn 4-face.



4-face-inside

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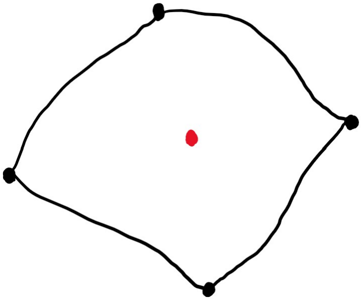

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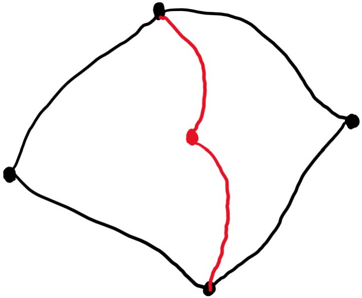

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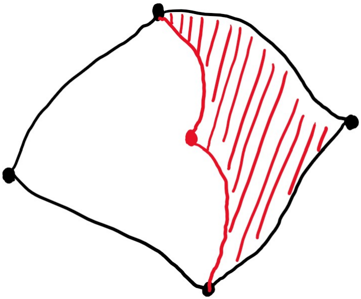

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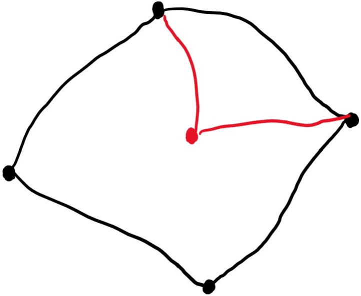

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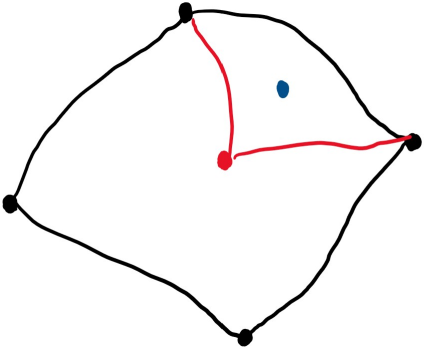

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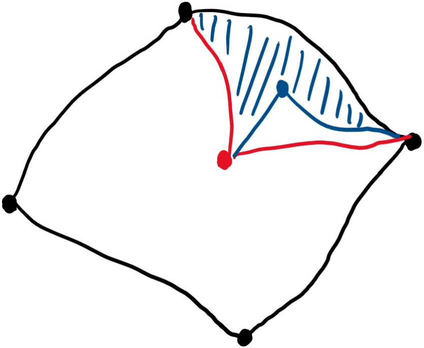

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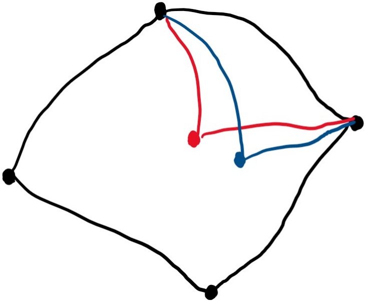

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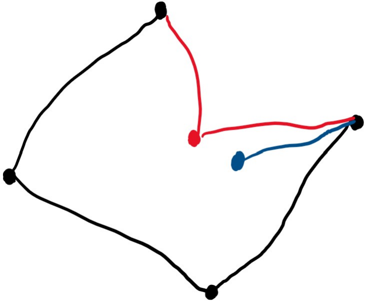

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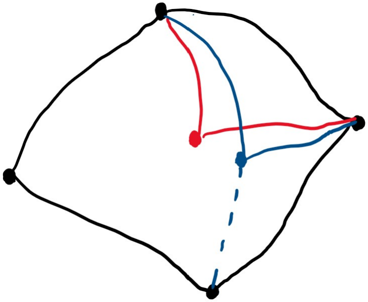

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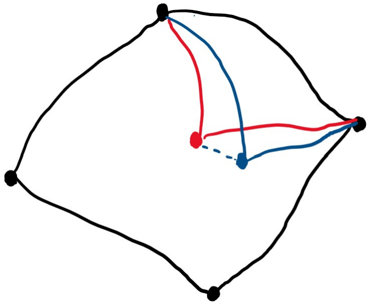


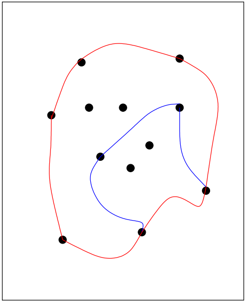
The claim is true!

4-face-inside Lemma

Lemma (Hubard–Suk 2023)

In a complete simple topological graph, suppose C is a plane k-cycle, if the face F enclosed by C contains at least $6k$ vertices in its interior, then there is a 4-face that lies inside F.

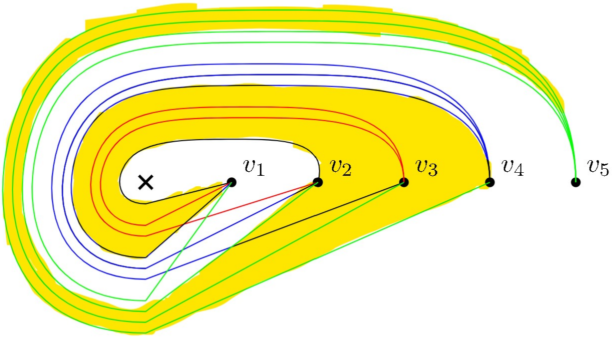
K_n =



4-face-inside Lemma

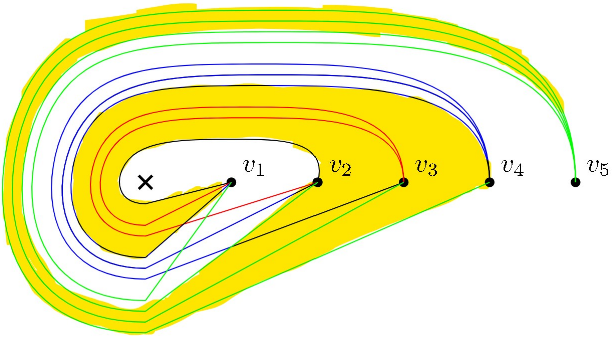
Remark

There is no 3-face-inside lemma!



Remark

There is no 3-face-inside lemma!



Problem

Is there a 6-face-inside lemma?

Other consequences

The key phenomenon has many other consequences.

Theorem (Fulek–Ruiz-Vargas 2013)

Every n-vertex complete simple topological graph contains at least $2n/3$ empty triangles.

Lemma (García–Pilz–Tejel 2021)

In a complete simple topological graph, every plane subgraph is contained in another plane subgraph that is biconnected.

Theorem (Aichholzer et al. 2022)

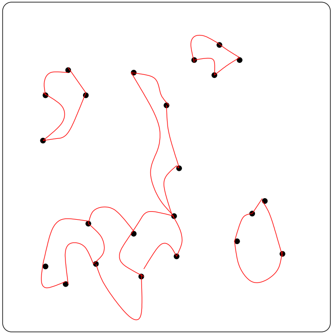
Every n-vertex complete simple topological graph contains at least $\Omega(n^{1/2})$ pairwise disjoint edges.

Disjoint 4-faces

Theorem (Z. 2023)

Every n-vertex complete simple topological graph generates at least $\Omega(n)$ pairwise disjoint 4-faces.

$$K_n =$$



Proof Sketch

1. Consider collections of pairwise disjoint “4-cells” (bounded or unbounded cells enclosed by 4-cycles).

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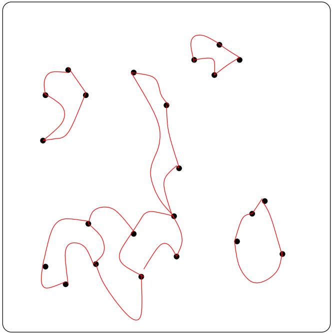
2. Fix a collection $\mathcal{C}$ that is largest and finest ($\mathcal{C}'$ is finer than $\mathcal{C}$ if any $c' \in \mathcal{C}'$ is contained in some $c \in \mathcal{C}$).
3. Let $H =$ vertices and edges on the boundaries of the 4-cells in $\mathcal{C}$.

$K_n =$ 

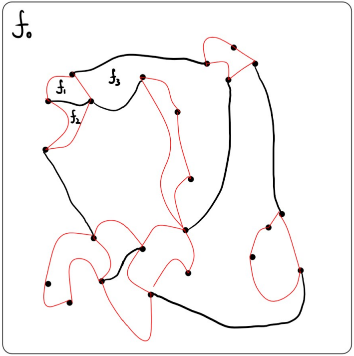
Proof Sketch

Lemma (García–Pilz–Tejel 2021)

In a complete simple topological graph, every plane subgraph is contained in another plane subgraph that is biconnected.

4. Construct a biconnected plane graph H' by only adding edges to H. Name the cells cut out by H' as $f_0, f_1, \ldots, f_k$.

$$K_n =$$



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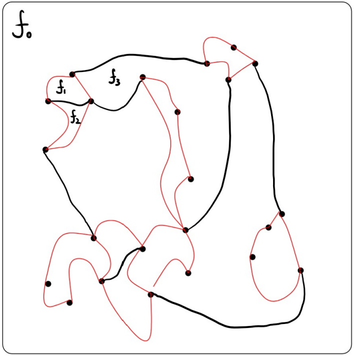
$K_n =$ 

5. Maximality of $\mathcal{C}$ + 4-face-inside Lemma:

$$\# \text{ of vertices inside of } f_i < 6 \cdot \# \text{ of boundary vertices of } f_i$$
$$v(f_i) < 6 \cdot |f_i|$$

Proof Sketch

7. Majority of vertices of G is on the boundary: $v(H) > \Omega(n)$.
   - $v(f_i) < 6|f_i|$
   - $\sum_i (v(f_i) + |f_i|) \geq n$
   - $\sum_i |f_i| = 2e(H')$ and $e(H') \leq 3v(H') - 6$
   - $v(H') = v(H)$

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8. Vertices of H comes from 4-cells: $|\mathcal{C}| \geq v(H)/4 > \Omega(n)$.
9. At most one cell in $\mathcal{C}$ is unbounded, the rest are pairwise disjoint 4-faces.

Bound from another side

Observation (Z. 2023)

For every $n \geq 1$, there is a n-vertex complete simple topological graph drawn in the unit square such that every face it generates has area at least $\Omega(1/n)$.

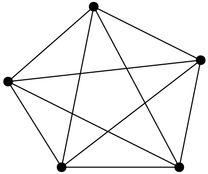

Figure: Convex geometric graph C_5.

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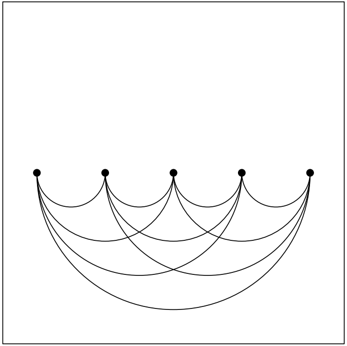

Figure: Convex geometric graph C_5.

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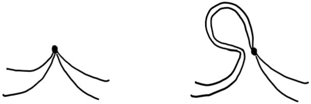

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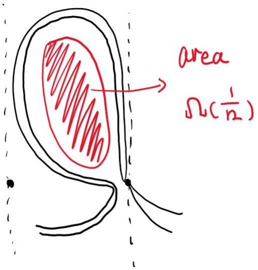

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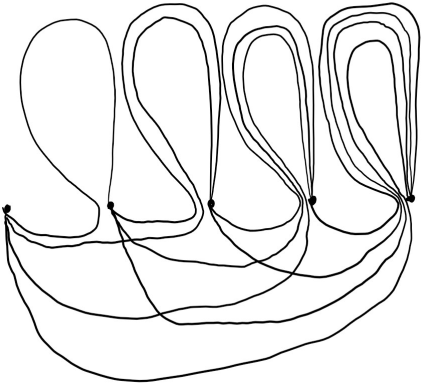

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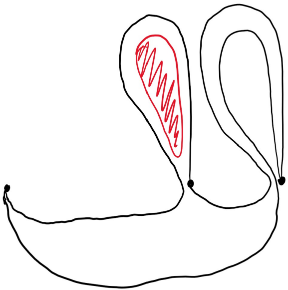
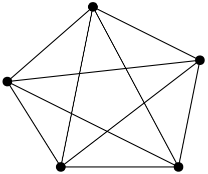
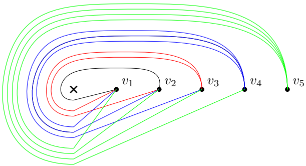


Disjoint k-faces

Theorem (Z. 2023)

For even $k \geq 4$, every n-vertex complete simple topological graph generates at least $(\log n)^{1/4-o(1)}$ pairwise disjoint k-faces.





Theorem (Suk–Z. 2022)

Every n-vertex complete simple topological graph has an induced subgraph on $m \geq (\log n)^{1/4-o(1)}$ vertices that is weakly isomorphic to either C_m or T_m.

Weak vs. Strong isomorphism

Weak isomorphism: graph isomorphism + crossing preserving.
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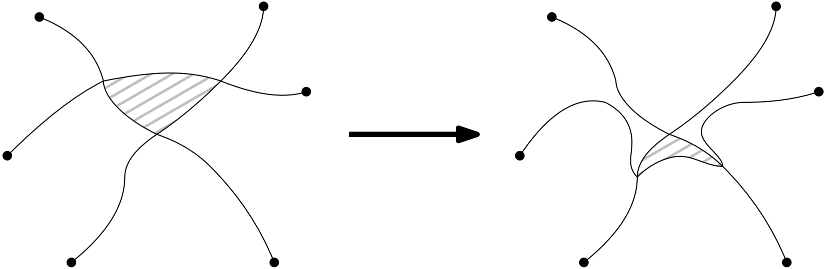

Figure: Triangle mutations can change the strong-isomorphism class.

Disjoint k-faces

Theorem (Gioan 2005/2022)

Two weakly isomorphic complete simple topological graphs are strongly isomorphic after a finite sequence of triangle mutations.

Observation: Triangle mutations preserve pairwise disjoint k-faces. Homeomorphisms of the sphere "almost" preserve pairwise disjoint k-faces.

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Theorem (Suk–Z. 2022)

Every n-vertex complete simple topological graph has an induced subgraph on $m \geq (\log n)^{1/4-o(1)}$ *vertices that is weakly isomorphic to either* C_m *or* T_m.

Fact: Either C_m or T_m has $\Omega(m)$ pairwise disjoint k-faces, for even k.

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Problem

Can we improve this lower bound to n^c for some $c = c(k) > 0$?

Problem

Is there a 6-face-inside lemma?

Thank you!!!

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